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Solitary internal waves with oscillatory tails

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Solitary internal waves in a density-stratified fluid of shallow depth are considered. According to the classical weakly nonlinear long-wave theory, the propagation of each long-wave mode is governed by the Korteweg–de Vries equation to leading order, and locally confined solitary waves with a 'sech^2' profile are possible. Using a singular-perturbation procedure, it is shown that, in general, solitary waves of mode $n > 1$ actually develop oscillatory tails of infinite extent, consisting of lower-mode short waves. The amplitude of these tails is exponentially small with respect to an amplitude parameter, and lies beyond all orders of the usual long-wave expansion. To illustrate the theory, two special cases of stratification are discussed in detail, and the amplitude of the oscillations at the solitary-wave tails is determined explicitly. The theoretical predictions are supported by experimental observations.

1. Introduction

The Korteweg–de Vries (KdV) equation describes the propagation of small-amplitude long waves in a wide variety of physical systems involving shallow fluids (see, for example, Benney 1966). It combines the leading-order nonlinear and dispersive effects, and predicts the existence of solitary-wave solutions that represent either elevation or depression waves with a 'sech^2' profile. In the classical case of gravity surface waves on water of finite depth, the KdV equation indicates that elevation solitary waves are possible. Such locally confined waves of permanent form indeed have been shown to exist by both accurate numerical computations (Longuet-Higgins & Fenton 1974; Byatt-Smith & Longuet-Higgins 1976) and rigorous analytical work (Amick & Toland 1981), based on the full nonlinear water-wave equations.

However, it is now recognized that the predictions of the KdV equation for gravity–capillary solitary waves are not entirely consistent with the full water-wave theory. According to the KdV equation, elevation solitary waves exist for low surface tension (surface-tension parameter, τ, less than $\frac{1}{3}$), while depression solitary waves are possible when $\tau > \frac{1}{3}$. On the other hand, the numerical computations of Hunter & Vanden-Broeck (1983), using the exact nonlinear equations, confirmed the existence of depression solitary waves for high surface tension ($\tau > \frac{1}{3}$), but could not find localized elevation waves of permanent form for $0 < \tau < \frac{1}{3}$. Rather, in this low-surface-tension regime, Hunter & Vanden-Broeck (1983) (see also Vanden-Broeck 1991) found symmetric periodic waves of permanent form with a single main hump, resembling a KdV solitary wave, which, however, was accompanied by short-wave oscillatory tails. These numerical results are supported by recent existence proofs of solitary waves with capillary ripples at infinity (Beale 1991; Sun 1991).

The reason for the appearance of short-wave tails when $0 < \tau < \frac{1}{3}$ can be readily understood on physical grounds. Gravity solitary waves are supercritical – their speed is higher than the linear-long-wave speed, which is the maximum phase speed of small-amplitude periodic waves if surface tension is neglected altogether ($\tau = 0$). Hence, it is not possible for small-amplitude short-wavelength gravity waves to move with a gravity solitary wave, and indeed no oscillatory tails are found. The same is true in the presence of high surface tension ($\tau > \frac{1}{3}$) since now solitary waves are subcritical and all small-amplitude periodic waves are supercritical. On the other hand, when $0 < \tau < \frac{1}{3}$, elevation solitary waves are supercritical, but, in this case, there exist small-amplitude short-wavelength capillary waves that can travel with the same phase speed; evidently, these short waves form the oscillatory tails found by Hunter & Vanden-Broeck (1983).

The numerical findings of Hunter & Vanden-Broeck (1983) point to the fact that the KdV equation is not a valid model when $0 < \tau < \frac{1}{3}$ because it is derived on the assumption that only long waves are present, and hence does not take into account the short lengthscale of oscillatory tails. For this reason, Hunter & Scheurle (1988) modified the KdV equation by adding a small fifth-order-derivative term. On the basis of this fifth-order KdV equation, they proved the existence of travelling-wave solutions in the form of slightly perturbed KdV solitary waves with small-amplitude oscillations at the tails. Independently, Pomeau, Ramani & Grammaticos (1988) considered the same model equation and studied the effect of the small fifth-order term on a KdV solitary wave, using singular-perturbation methods. From a theoretical point of view, this proves to be a very interesting problem; the amplitude of the oscillatory tails turns out to be exponentially small, so that it would not appear at any order in an expansion in powers of the small parameter multiplying the fifth-order term of the model equation. To calculate the amplitude of the tails, Pomeau *et al.* (1988) used a nonlinear WKB technique devised earlier by Segur & Kruskal (1987) (see also Kruskal & Segur 1991). The results of the perturbation theory have been confirmed in a recent numerical study of the fifth-order KdV equation (Boyd 1991).

The present paper is concerned with solitary waves in a density-stratified fluid of shallow depth bounded by rigid walls. In this case, it is well known that there exists an infinite set of linear internal-wave modes. For each of these modes, the propagation of long waves is governed by a KdV equation to leading order, according to the classical long-wave expansion (Benney 1966), suggesting that solitary waves are possible. So the question arises as to whether internal solitary waves develop short-wave oscillatory tails, as in the analogous problem of gravity–capillary surface waves discussed earlier. It follows from the KdV theory that an internal solitary wave of a certain mode is supercritical with respect to the corresponding linear-long-wave speed, which is the maximum phase speed of all linear periodic waves of this mode (Yih 1979, §4.1.4); therefore, it is expected that small-amplitude waves associated with the same mode as a KdV solitary wave cannot form oscillatory tails. However, the linear phase speed of waves of a certain wavenumber decreases as the mode number increases (Yih 1979, §4.1.2), implying that, for a solitary wave of mode higher than the first, there exist lower-mode linear short waves that can move with the same phase speed (see figure 1). This suggests that a KdV solitary wave corresponding to a mode higher than the first can develop oscillations at its tails owing to lower-mode short waves. Note that a similar situation arises in the problem of equatorial Rossby solitary waves (Boyd 1989).

In accordance with the above qualitative remarks, an asymptotic theory is

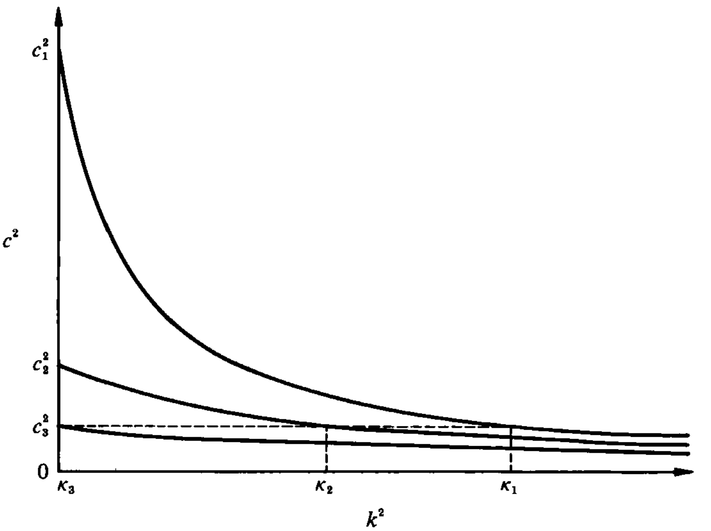

FIGURE 1. Variation of c^2 (the square of the linear-wave speed) with k^2 (the square of the wavenumber) for the first three linear internal-wave modes, in a typical situation. Mode-1 waves with wavenumber $\kappa_1^{\frac{1}{2}}$ and mode-2 waves with wavenumber $\kappa_2^{\frac{1}{2}}$ have phase speed equal to the long-wave speed of mode-3 waves.

presented here to establish the presence and calculate the amplitude of short-wave oscillations at the tails of KdV solitary internal waves of mode higher than the first. For this purpose, matched asymptotic expansions and the Borel technique for summing divergent series are used as in Pomeau *et al.* (1988), but the analysis is more involved than in the corresponding problem of the fifth-order KdV equation. As expected, the amplitude of the oscillatory tails is exponentially small, and is determined by asymptotic matching of the usual long-wave (outer) expansion with inner expansions valid near the singularities of the outer expansion. However, the inner equations are essentially the fully nonlinear internal-wave equations, and, in general, the amplitude of the tails depends on all nonlinear and dispersive terms; so it does not seem possible that the correct magnitude of the oscillatory tails can be obtained from model equations alone. To illustrate the theory, two particular examples, where the nonlinear terms of the governing equations are only quadratic, are discussed in detail. In the first case, the square of the Brunt–Väisälä frequency is taken to be constant, while in the second case it is assumed to vary linearly with depth and the Boussinesq approximation is made.

It is interesting to note that the appearance of short-wave oscillations at the tails of internal solitary waves of mode higher than the first is supported by experimental observations. In the course of their investigation of internal-wave disturbances, generated by stratified flow over a sill, Farmer & Smith (1980) observed mode-2 'solitary-like' waves followed by a train of smaller-amplitude mode-1 short waves (see figure 2†), in qualitative agreement with the theoretical predictions. Also, in laboratory experiments, Davis & Acrivos (1967) noted that large-amplitude solitary waves in deep fluids shed oscillatory waves downstream that caused radiation damping of the main disturbance, consistent with the conclusions reached here (see §5).

† This particular figure does not appear in Farmer & Smith (1980); it was kindly provided to us by Dr Farmer.

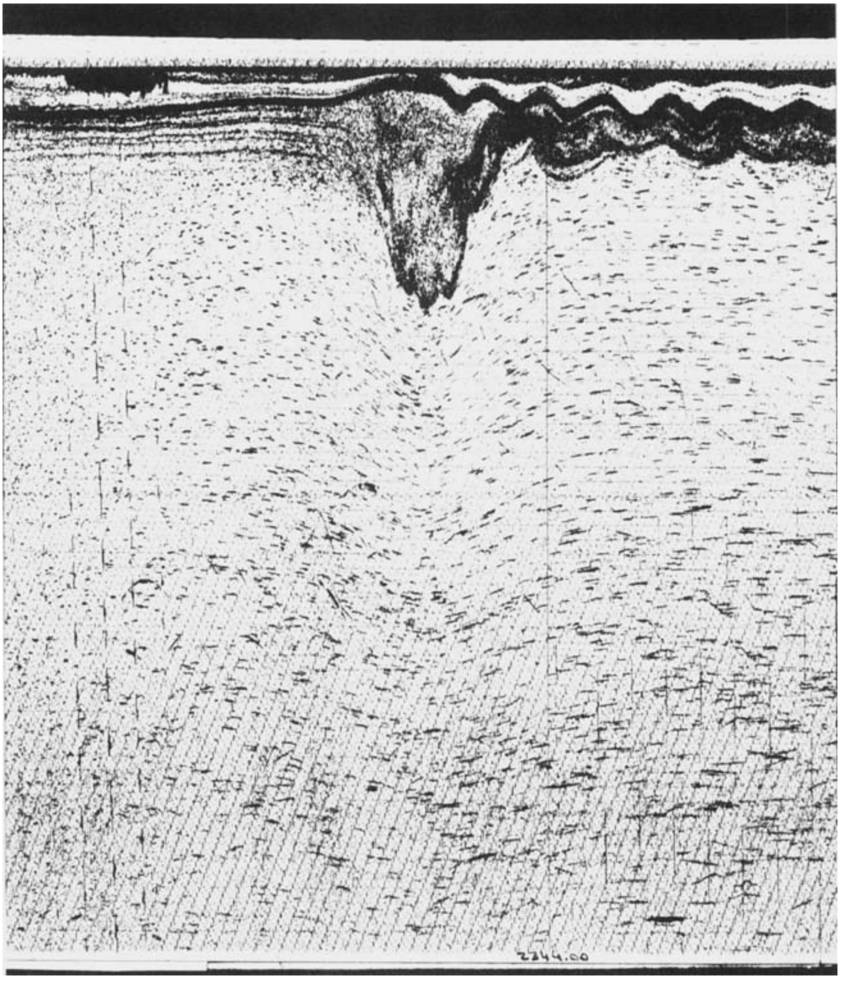

FIGURE 2. Acoustical image of internal-wave disturbances generated by stratified flow past a sill in the field experiments of Farmer & Smith (1980). The streamlines indicate that the main disturbance is a mode-2 solitary-like wave and is followed by a train of smaller-amplitude mode-1 short waves.

Recently, Turkington, Eydeland & Wang (1991), using a variational formulation of the governing equations, proposed a numerical technique for computing solitary-wave solutions in a stratified fluid, and presented several examples of mode-1 solitary waves; as expected, these waves are locally confined. In earlier related work, Tung, Chan & Kubota (1982) proved analytically and confirmed through numerical computations that large-amplitude locally confined mode-1 and mode-2 solitary waves are possible in a stratified fluid of finite depth, under the Boussinesq approximation. However, in discussing mode-2 solitary waves, they further assume that the density stratification is such that the Brunt–Väisälä frequency is symmetric about the fluid-layer centreline. This additional condition precludes the appearance of mode-1 oscillatory tails because waves of the first mode are symmetric while waves of the second mode are antisymmetric about the centreline. Nevertheless, mode-1 oscillations are still expected to develop at the tails of mode-3 solitary waves, which are also symmetric, but Tung *et al.* (1982) do not report calculations of solitary waves of mode-3 or higher.

2. Preliminaries

Consider two-dimensional internal-wave disturbances in an inviscid, incompressible layer of finite depth. Dimensionless variables will be used throughout, based on the fluid depth as the lengthscale and the inverse of a typical value of the Brunt–Väisälä frequency as the timescale. In terms of the undisturbed fluid density $\rho_0(z)$, the Brunt–Väisälä frequency $N(z)$ is defined by

$$\beta\rho_0 N^2 = -\rho_{0z}, \tag{1}$$

β being the Boussinesq parameter.

As we are interested in travelling waves of permanent form, it is convenient to adopt a reference frame moving with the wave speed c so that the flow is made steady, and introduce the stream function $\Psi = c(z+\psi)$ where $\psi(x,z)$ describes the wave disturbance. In terms of Ψ, the horizontal and vertical velocity components are given by $-\Psi_z, \Psi_x$ respectively, and, thus, the incompressibility condition is automatically satisfied.

Using the fact that $\rho \rightarrow \rho_0(z)$ in the small-amplitude limit ($\psi \rightarrow 0$), the equation for conservation of mass can be readily integrated to determine the density:

$$\rho = \rho_0(\Psi/c). \tag{2}$$

The equation governing ψ is obtained by first eliminating the pressure from the two momentum equations. Using (1), (2), the resulting equation then can be manipulated to the standard form (see, for example, Yih 1979, §4.2)

$$J\left(\nabla^2\psi + N^2(z+\psi)\left\{\frac{\psi}{c^2} - \frac{\beta}{2c^2}(\Psi_x^2 + \Psi_z^2)\right\}, \Psi\right) = 0, \tag{3}$$

where $J(a,b)$ stands for the Jacobian $a_x b_z - a_z b_x$. Equation (3) can be integrated once to give

$$\nabla^2\psi + N^2(z+\psi)\left\{\frac{\psi}{c^2} - \frac{\beta}{2c^2}(\Psi_x^2 + \Psi_z^2)\right\} = H(\Psi), \tag{4}$$

where H is an arbitrary function. To specify H, note again that (4) must remain valid in the small-amplitude limit ($\psi \rightarrow 0$), in which case

$$H(cz) = -\tfrac{1}{2}\beta N^2(z);$$

hence

$$H(\Psi) = -\tfrac{1}{2}\beta N^2(\Psi/c),$$

and (4) becomes

$$\nabla^2\psi + N^2(z+\psi)\{\lambda\psi - \beta\psi_z - \tfrac{1}{2}\beta(\psi_x^2+\psi_z^2)\} = 0 \quad (-\infty < x < \infty, 0 \leqslant z \leqslant 1), \tag{5a}$$

with

$$\lambda = 1/c^2. \tag{5b}$$

To complete the formulation of the problem, we need to specify boundary conditions. For simplicity, it will be assumed that the fluid layer is bounded by rigid walls, so that

$$\psi = 0 \quad (z = 0, 1). \tag{6}$$

In the small-amplitude limit ($\psi \rightarrow 0$), (5a) can be linearized to leading order. Linear wave modes of the form $\psi = \phi(z)\exp(ikx)$ then satisfy

$$(\rho_0\phi_z)_z + \rho_0(\lambda N^2(z) - k^2)\phi = 0 \quad (0 \leqslant z \leqslant 1), \tag{7a}$$

$$\phi = 0 \quad (z = 0, 1). \tag{7b}$$

This may be viewed as an eigenvalue problem, the eigenvalue parameter being either λ for given wavenumber k, or k^2 for fixed λ. In particular, the long-wave modes $\{f_s(z), \lambda_s\}$ $(s = 1, 2, \ldots)$ correspond to $k = 0$ and are defined by the eigenvalue problem

$$(\rho_0 f_{sz})_z + \lambda_s \rho_0 N^2 f_s = 0 \quad (0 \leqslant z \leqslant 1), \tag{8a}$$

$$f_s = 0 \quad (z = 0, 1). \tag{8b}$$

The eigenvalues $0 < \lambda_1 < \lambda_2 < \ldots$ are related to the long-wave speeds $c_1 > c_2 > \ldots > 0$ through (5*b*); the eigenmodes form an orthogonal and complete set with the orthogonality relation

$$\int_0^1 \rho_0 N^2 f_r f_s \,\mathrm{d}z = I_s \delta_{rs}, \tag{9}$$

where I_s are normalization constants and δ_{rs} denotes the Kronecker delta.

It will also prove useful for the subsequent discussion to define another set of modes $\{\phi_r(z), \kappa_r\}$ $(r = 1, 2, \ldots)$ by taking c to be equal to one of the long-wave speeds c_n, say, so that $\lambda = \lambda_n$ in (7*a*), and considering $\kappa \equiv k^2$ to be the eigenvalue parameter:

$$(\rho_0 \phi_{rz})_z + \rho_0(\lambda_n N^2 - \kappa_r)\phi_r = 0 \quad (0 \leqslant z \leqslant 1), \tag{10a}$$

$$\phi_r = 0 \quad (z = 0, 1). \tag{10b}$$

These modes also form an orthogonal and complete set with orthogonality relation

$$\int_0^1 \rho_0 \phi_r \phi_s \,\mathrm{d}z = K_r \delta_{rs}, \tag{11}$$

K_r being normalization constants. Physically, this new set of modes comprises waves with wavenumber $\kappa_r^{\frac{1}{2}}$ and phase speed c_n. So, the long-wave mode $f_n(z)$ clearly is an eigenmode with eigenvalue $\kappa_n = 0$. In addition, however, it is known (Yih 1979, §4.1.3) that there are $n-1$ positive eigenvalues $\kappa_1 > \kappa_2 > \ldots \kappa_{n-1} > \kappa_n = 0$ that correspond to short waves moving with the long-wave speed c_n. It is clear (see figure 1) that such waves exist only if c_n is the long-wave speed of a mode higher than the first $(n > 1)$, and, as will be demonstrated in §5, they can give rise to oscillations at the tails of a KdV solitary wave of mode n. The rest of the eigenvalues of (10) are negative $(\kappa_r < 0, r > n)$, and correspond to evanescent disturbances.

3. Long-wave expansion

In the classical weakly nonlinear long-wave theory, it is assumed that the horizontal lengthscale of wave disturbances is long compared with the fluid depth, so that the scaled horizontal coordinate $X = \epsilon x$ $(0 < \epsilon \ll 1)$ is appropriate. Furthermore, to balance dispersive with nonlinear effects, the disturbance amplitude is taken to be $O(\epsilon^2)$. So, returning to the governing equation (5*a*), $\psi(X, z; \epsilon^2)$ is expanded as

$$\psi = \epsilon^2(\psi_0 + \epsilon^2\psi_1 + \epsilon^4\psi_2 + \ldots), \tag{12}$$

and λ is chosen such that the wave speed c is close to c_n, the linear-long-wave speed of mode n, say:

$$\lambda = \lambda_n - \alpha\epsilon^2, \quad \alpha = \alpha_0 + \epsilon^2\alpha_1 + \ldots. \tag{13}$$

In view of (12), the nonlinear term $N^2(z+\psi)$ in (5*a*) is also expanded in a Taylor series:

$$N^2(z+\psi) = N^2(z) + \sum_{j=1}^{\infty} M_j(z)\psi^j, \tag{14}$$

where $$M_j \equiv \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}z^j}N^2(z).$$

Upon substitution of (12)–(14) into (5a), (6), it is found that ψ_0 satisfies

$$(\rho_0 \psi_{0z})_z + \lambda_n \rho_0 N^2 \psi_0 = 0 \quad (0 \leqslant z \leqslant 1),$$

subject to the boundary conditions $\psi_0 = 0$ at $z = 0, 1$. Therefore

$$\psi_0 = A(X) f_n(z), \tag{15}$$

where $f_n(z)$ is the long-wave mode corresponding to $\lambda = \lambda_n$, defined in (8), and $A(X)$ is an as yet undetermined amplitude function. Proceeding to $O(\epsilon^4)$, ψ_1 satisfies the inhomogeneous problem

$$(\rho_0 \psi_{1z})_z + \lambda_n \rho_0 N^2 \psi_1 + \rho_0 F_1 = 0 \quad (0 \leqslant z \leqslant 1), \tag{16a}$$

$$\psi_1 = 0 \quad (z = 0, 1), \tag{16b}$$

where $$F_1 = \psi_{0XX} - N^2(\alpha_0 \psi_0 + \tfrac{1}{2}\beta \psi_{0z}^2) + M_1(\lambda_n \psi_0^2 - \beta \psi_0 \psi_{0z}).$$

Invoking the usual orthogonality argument, for this problem to have a solution it is necessary that the solvability condition

$$\int_0^1 \rho_0 F_1 f_n \,\mathrm{d}z = 0 \tag{17}$$

is met, implying that $A(X)$ has to satisfy

$$\delta A_{XX} - \alpha_0 A - \tfrac{1}{2}\mu A^2 = 0 \tag{18}$$

with $$I_n \delta = \int_0^1 \rho_0 f_n^2 \,\mathrm{d}z,$$

$$I_n \mu = -2\int_0^1 \rho_0 f_n (\lambda_n M_1 f_n^2 - \beta M_1 f_n f_n' - \tfrac{1}{2}\beta N^2 f_n'^2)\,\mathrm{d}z.$$

Equation (18) is the steady KdV equation and the solution of interest here is the KdV solitary wave:

$$A = a\,\mathrm{sech}^2 \gamma X \tag{19a}$$

with $$\alpha_0 = -\tfrac{1}{3}\mu a = 4\delta\gamma^2, \tag{19b}$$

and $\gamma > 0$, say. Depending on the sign of μ, this is a wave of elevation ($\mu < 0$, $a > 0$) or depression ($\mu > 0, a < 0$) that moves with speed slightly higher than the corresponding linear-long-wave speed ($\alpha_0 > 0$ in (13)).

According to (19), each internal-wave mode can support locally confined solitary waves; this is the well-known result of the standard small-amplitude long-wave theory. However, bearing in mind the earlier work of Segur & Kruskal (1987) and Pomeau *et al.* (1988), it is important to note that the solution (19) is singular in the complex X-plane at $X = \pm X_p$, where $\gamma X_p = \tfrac{1}{2}\mathrm{i}(2p+1)\pi$ ($p = 0, 1, 2, \ldots$). Near each of these singularities, the long-wave theory is expected to break down and an inner expansion is necessary. Matching of the long-wave expansion (12) with these inner expansions determines whether short-wave oscillations appear at the tails of solitary waves.

To identify the appropriate inner scales of the expansions near the singularities, we proceed to find higher-order terms in the long-wave expansion (12). Having determined A, the inhomogeneous term in (16a) is now known:

$$F_1 = \alpha_0\left(\frac{1}{\delta} - N^2\right) f_n A + E_1 A^2,$$

where

$$E_1(z) = \frac{\mu}{2\delta} f_n + M_1(\lambda_n f_n^2 - \beta f_n f_n') - \tfrac{1}{2}\beta N^2 f_n'^2, \tag{20}$$

and ψ_1 takes the form

$$\psi_1 = p_1 A + q_1 A^2 + f_n A_1. \tag{21}$$

Here $p_1(z), q_1(z)$ satisfy the inhomogeneous problems

$$(\rho_0 p_{1z})_z + \lambda_n \rho_0 N^2 p_1 + \alpha_0 \rho_0 \left(\frac{1}{\delta} - N^2\right) f_n = 0 \quad (0 \leqslant z \leqslant 1), \tag{22a}$$

$$p_1 = 0 \quad (z = 0, 1), \tag{22b}$$

and

$$(\rho_0 q_{1z})_z + \lambda_n \rho_0 N^2 q_1 + \rho_0 E_1 = 0 \quad (0 \leqslant z \leqslant 1), \tag{23a}$$

$$q_1 = 0 \quad (z = 0, 1). \tag{23b}$$

In view of the compatibility condition (17), both of these problems have solutions, which can be made unique by requiring that

$$\int_0^1 \rho_0 N^2 p_1 f_n \,\mathrm{d}z = 0, \quad \int_0^1 \rho_0 N^2 q_1 f_n \,\mathrm{d}z = 0. \tag{24a, b}$$

The amplitude $A_1(X)$ in (21) is determined by considering the $O(\epsilon^6)$ problem for ψ_2:

$$(\rho_0 \psi_{2z})_z + \lambda_n \rho_0 N^2 \psi_2 + \rho_0 F_2 = 0 \quad (0 \leqslant z \leqslant 1),$$
$$\psi_2 = 0 \quad (z = 0, 1),$$

where

$$F_2 = \psi_{1XX} - N^2(\alpha_0 \psi_1 + \alpha_1 \psi_0 + \beta \psi_{0z} \psi_{1z} + \tfrac{1}{2}\beta \psi_{0X}^2)$$
$$+ M_1(2\lambda_n \psi_0 \psi_1 - \alpha_0 \psi_0^2 - \beta(\psi_0 \psi_1)_z - \tfrac{1}{2}\beta \psi_0 \psi_{0z}^2) + M_2(\lambda_n \psi_0^3 - \beta \psi_0^2 \psi_{0z}).$$

Imposing the compatibility condition

$$\int_0^1 \rho_0 F_2 f_n \,\mathrm{d}z = 0,$$

taking into account (21), (24) and making further use of (18), then yields

$$\delta A_{1XX} - \alpha_0 A_1 - \mu A A_1 + \eta_1 A + \zeta_1 A^2 + \sigma_1 A^3 = 0, \tag{25}$$

where

$$I_n \eta_1 = -I_n \alpha_1 + \alpha_0 \int_0^1 \rho_0 f_n p_1 \left(\frac{1}{\delta} - N^2\right) \mathrm{d}z,$$

$$I_n \zeta_1 = \int_0^1 \rho_0 f_n \left\{\frac{\mu}{2\delta} p_1 + 4\frac{\alpha_0}{\delta} q_1 - \beta N^2 \left(f_n' p_1' + \frac{\alpha_0}{2\delta} f_n^2\right)\right\} \mathrm{d}z$$
$$+ \int_0^1 \rho_0 f_n M_1 (2\lambda_n f_n p_1 - \alpha_0 f_n^2 - \beta (f_n p_1)') \,\mathrm{d}z,$$

$$I_n \sigma_1 = \int_0^1 \rho_0 f_n \left\{\frac{5\mu}{3\delta} q_1 - \beta N^2 \left(f_n' q_1' + \frac{\mu}{6\delta} f_n^2\right)\right\} \mathrm{d}z$$
$$+ \int_0^1 \rho_0 f_n \{M_1(2\lambda_n f_n q_1 - \tfrac{1}{2}\beta f_n f_n'^2 - \beta(f_n q_1)') + M_2(\lambda_n f_n^3 - \beta f_n^2 f_n')\} \,\mathrm{d}z.$$

Note that the regular homogeneous solution of (25) behaves like

$$\exp\left\{-\left(\frac{\alpha_0}{\delta}\right)^{\frac{1}{2}}|X|\right\} \quad (|X|\to\infty),$$

which, in view of (19), is identical with the asymptotic behaviour of $A(X)$. Hence, to avoid secular behaviour of A_1 at the tails of the KdV solitary wave, we require $\eta_1 = 0$ in (25); this condition, together with (24*a*), specifies α_1:

$$I_n\,\alpha_1 = \frac{\alpha_0}{\delta}\int_0^1 \rho_0 f_n p_1\,\mathrm{d}z.$$

Using (18), the solution of (25) for A_1 then is found to be

$$A_1 = \theta_1 A + \nu_1 A^2, \tag{26}$$

where
$$\tfrac{2}{3}\mu\nu_1 + \sigma_1 = 0, \quad \mu\theta_1 = 2\zeta_1 + 6\alpha_0\nu_1. \tag{27 a, b}$$

This completes the solution for ψ_1. Combining (15), (21), (26), and (27), the results of the long-wave expansion (12), correct to $O(\epsilon^4)$, can be summarized as

$$\psi = \epsilon^2 A f_n + \epsilon^4\{p_1 A + q_1 A^2 + (\theta_1 A + \nu_1 A^2) f_n\} + O(\epsilon^6). \tag{28}$$

It is clear from (28) that the $O(\epsilon^4)$ corrections to the KdV solitary wave (15), (19) remain locally confined. Moreover, by carrying out the expansion (12) to higher order, it is straightforward to confirm that this is the case for all terms in (12), in agreement with the fact (see §5) that the amplitude of possible oscillatory tails is exponentially small.

On the other hand, as expected, expansion (28) becomes disordered near the singularities of ψ_0 at $X = \pm X_p$. More specifically, $A(X)$ has double-pole singularities at $X = \pm X_p$ and the $O(\epsilon^4)$ term becomes comparable in magnitude with the $O(\epsilon^2)$ term when $X \pm X_p = O(\epsilon)$. This suggests the definition of a new (inner) variable ξ, appropriate near $X = X_0$:

$$X = \mathrm{i}\frac{\pi}{2\gamma} + \epsilon\xi. \tag{29}$$

In terms of ξ, the inner limit of the outer expansion (28) is

$$\psi \sim \left\{-\frac{a}{\gamma^2\xi^2} + \nu_1\frac{a^2}{(\gamma^2\xi^2)^2} + \dots\right\} f_n + \left\{\frac{a^2}{(\gamma^2\xi^2)^2} q_1 + \dots\right\} + O(\epsilon^2), \tag{30}$$

indicating that $\psi = O(1)$ near the singularity. Expressions similar to (30) hold near the rest of the singularities. However, it turns out that the two singularities closest to the real X-axis at $X = \pm X_0$ make the dominant contribution to the oscillatory tails, and in the following sections we shall discuss the inner solutions near these singularities.

4. Inner problem

Attention is now focused on the inner problem near the singularity of the long-wave expansion at $X = X_0$. In terms of the inner variable ξ defined in (29), taking into account (13), the governing equation (5*a*) for the inner solution $\psi(\xi, z)$ becomes to leading order

$$\psi_{zz} + \psi_{\xi\xi} + N^2(z+\psi)\{\lambda_n\psi - \beta\psi_z - \tfrac{1}{2}\beta(\psi_\xi^2 + \psi_z^2)\} = 0. \tag{31}$$

This equation must be solved subject to the boundary conditions (6) and the matching condition (30) in the intermediate region $1 \ll |\xi| \ll 1/\epsilon, \operatorname{Im}\xi < 0$.

As expected – and will be verified in detail in §4.1 – expansion (30) is an asymptotic solution of (31), subject to (6), in the intermediate matching region. It is important to note, however, that the modes defined earlier in (10) also satisfy the linearized version of (31), subject to (6). Hence, assuming that the KdV solitary wave (15), (19) is associated with a long-wave mode higher than the first ($n > 1$) so that some of the modes (10) are oscillatory, it is possible to add an exponentially small amount of these short-wave modes to the asymptotic solution (30), thus causing the tails to become oscillatory. Whether such oscillations actually appear cannot be decided on the basis of the long-wave expansion (12) alone. Rather, it would seem necessary to solve the inner equation (31), subject to (6), (30), assuming, say, that no oscillations are present at the left-hand tail ($\operatorname{Re}\xi \to -\infty$) in order to obtain the asymptotic form of ψ as $\operatorname{Re}\xi \to \infty$, and thereby determine the amplitude of possible oscillations at the right-hand tail. In simpler problems, where the inner equations are ordinary differential equations, this task has been carried out numerically (Kruskal & Segur 1991; Grimshaw & Hooper 1991); but here the inner equation (31) is a partial differential equation that includes all nonlinear and dispersive terms of the governing equation (5*a*), and a numerical approach would be rather impractical – essentially, it would amount to solving the original, fully nonlinear problem numerically. However, the Borel-summation technique, suggested by Pomeau *et al.* (1988) in connection with the fifth-order KdV equation, can be conveniently used to avoid a fully numerical treatment of the inner problem. The main idea is to 'sum' the (divergent) asymptotic series (30) using Borel summation (Bender & Orszag 1978, §8.2); the singularities of the resulting Borel sum in the (transformed) complex plane can then be related to exponentially small terms that lie beyond all orders in the asymptotic expansion (30). Following this procedure, the appearance of exponentially small oscillatory tails is established analytically, and the corresponding amplitude can be determined simply by solving a sequence of linear two-point boundary-value problems numerically.

4.1. *Asymptotic solution*

In preparation for the Borel summation of the asymptotic expansion (30), we now proceed to find expressions for the higher-order terms in this series. So, we write

$$\psi \sim \sum_{m=1} G_m(z)\,\xi^{-2m}, \tag{32}$$

and upon substitution into (31), we find

$$\sum_{m=1} \{G_m'' - \beta N^2 G_m' + \lambda_n N^2 G_m\}\,\xi^{-2m} + \sum_{m=1} 2m(2m+1)\,G_m\,\xi^{-2(m+1)}$$
$$+\lambda_n \sum_{j=1}^{\infty} M_j(z)\,\psi^{j+1} - \beta \sum_{j=1}^{\infty} \frac{1}{j+1} M_j(z)\,(\psi^{j+1})_z - \tfrac{1}{2}\beta \sum_{j=0}^{\infty} M_j(z)\,\psi^j(\psi_\xi^2 + \psi_z^2) = 0 \quad (0 \leqslant z \leqslant 1). \tag{33}$$

Also, in view of (6), $G_m(z)\,(m \geqslant 1)$ satisfy the boundary conditions

$$G_m = 0 \quad (z = 0, 1). \tag{34}$$

Equating coefficients of ξ^{-2m} to zero in (33) yields an infinite sequence of boundary-value problems for $G_m(z)$. In general, of course, the nonlinear terms in (33) give rise to rather cumbersome multiple convolution sums. To illustrate, suppose that the Taylor series (14) terminates at the linear term:

$$M_j(z) = 0 \quad (j \geqslant 2);$$

then, the resulting sequence of equations reads

$$G''_m - \beta N^2 G'_m + \lambda_n N^2 G_m + (2m-1)(2m-2)\, G_{m-1} + \lambda_n M_1 J_m - \tfrac{1}{2}\beta M_1 J'_m - \tfrac{1}{2}\beta N^2 (L_m + S_m) - \tfrac{1}{2}\beta M_1 \sum_{j=1}^{m-1} (L_{m-j} + S_{m-j})\, G_j = 0 \quad (0 \leqslant z \leqslant 1), \tag{35}$$

where

$$J_m = \sum_{j=1}^{m-1} G_{m-j} G_j \quad (m \geqslant 2), \quad J_1 = 0;$$

$$L_m = \sum_{j=1}^{m-1} G'_{m-j} G'_j \quad (m \geqslant 2), \quad L_1 = 0;$$

$$S_m = \sum_{j=1}^{m-2} 4j(m-j-1)\, G_{m-j-1} G_j \quad (m \geqslant 3), \quad S_1 = S_2 = 0.$$

It is now straightforward to verify that (30) satisfies the inner equation (31) asymptotically, subject to the boundary conditions (6). In particular, for $m = 1$, (35) yields

$$G''_1 - \beta N^2 G'_1 + \lambda_n N^2 G_1 = 0 \quad (0 \leqslant z \leqslant 1),$$

and, in view of (30), the appropriate solution that satisfies (34) is

$$G_1 = -\frac{a}{\gamma^2} f_n. \tag{36}$$

Similarly, for $m = 2$, one has

$$G''_2 - \beta N^2 G'_2 + \lambda_n N^2 G_2 + \frac{\alpha^2}{\gamma^4} E_1 = 0 \quad (0 \leqslant z \leqslant 1),$$

$$G_2 = 0 \quad (z = 0, 1),$$

and recalling (23),

$$G_2 = \frac{a^2}{\gamma^4} q_1(z) + \gamma_2 f_n(z), \tag{37}$$

where γ_2 is an as yet undetermined constant. Proceeding to $m = 3$, G_3 satisfies

$$G''_3 - \beta N^2 G'_3 + \lambda_n N^2 G_3 + R_3 = 0 \quad (0 \leqslant z \leqslant 1),$$

$$G_3 = 0 \quad (z = 0, 1),$$

where

$$R_3 = 20 G_2 - \beta N^2 (G'_1 G'_2 + 2G_1^2) + M_1 (2\lambda_n G_1 G_2 - \beta (G_1 G_2)' - \tfrac{1}{2}\beta G_1'^2 G_1).$$

Imposing the solvability condition

$$\int_0^1 \rho_0 R_3 f_n \, dz = 0,$$

after some manipulation using (27), (36), and (37), determines γ_2:

$$\gamma_2 = \frac{a^2}{\gamma^4} \nu_1.$$

Hence, the first two terms of the series (32) are consistent with (30).

In principle, one can calculate higher-order terms in the asymptotic solution (32), but the algebra becomes very tedious owing to the nonlinear terms in (35). However, in the limit $m \to \infty$, (35) simplifies considerably because the contribution of the nonlinear terms is subdominant. To verify this, suppose that the dominant balance in (35) as $m \to \infty$ is

$$G''_m - \beta N^2 G'_m + \lambda_n N^2 G_m \sim -(2m-1)(2m-2)\,G_{m-1} \quad (0 \leqslant z \leqslant 1), \tag{38}$$

subject to (34). Compatibility then requires

$$\int_0^1 \rho_0 G_{m-1} f_n \,\mathrm{d}z = 0,$$

and, recalling the orthogonality relation (11), this condition is automatically met if G_m is expressed in terms of the modes defined in (10), excluding the long-wave mode $\phi_n \equiv f_n$:

$$G_m(z) = \sum_{\substack{r=1\\ r\neq n}}^{\infty} g_{mr}\,\phi_r(z). \tag{39}$$

Substituting (39) into (38) then yields

$$\kappa_r g_{mr} \sim -(2m-1)(2m-2)\,g_{m-1,r},$$

where κ_r are the eigenvalues of the eigenvalue problem (10). Hence,

$$g_{mr} \sim C_r(-1)^m \kappa_r^{-m}(2m-1)! \quad (m \to \infty), \tag{40}$$

and (39) gives

$$G_m \sim (2m-1)! \sum_{\substack{r=1\\ r\neq n}}^{\infty} C_r(-1)^m \kappa_r^{-m}\phi_r(z) \quad (m \to \infty), \tag{41}$$

where C_r are constants that depend on $G_1(z)$ in (36) and cannot be determined by asymptotic analysis alone.

Clearly, the leading-order behaviour (41) is consistent with the assumed dominant balance (38). To find the next term in the expansion (41), we put

$$G_m = \hat{G}_m(2m-1)!, \tag{42}$$

and keeping the next-order terms in (35), one has

$$\hat{G}''_m - \beta N^2 \hat{G}'_m + \lambda_n N^2 \hat{G}_m + \hat{G}_{m-1} \sim \frac{1}{(2m-1)(2m-2)}\frac{a}{\gamma^2}\{2\lambda_n M_1 \hat{G}_{m-1} f_n - \beta M_1(\hat{G}_{m-1} f_n)' - \beta N^2 \hat{G}'_{m-1} f'_n\}. \tag{43}$$

Also, we write

$$\hat{G}_m = \sum_{\substack{r=1\\ r\neq n}}^{\infty} \hat{g}_{mr}\phi_r + \hat{g}_m f_n, \tag{44}$$

where, according to (40),

$$\hat{g}_{mr} \sim C_r(-1)^m \kappa_r^{-m} \quad (m \to \infty). \tag{45}$$

Imposing now compatibility in (43), using (45), yields

$$\hat{g}_m \sim \frac{12}{\mu}\frac{1}{2m(2m+1)} \sum_{\substack{r=1\\ r\neq n}}^{\infty} C_r(-1)^m \kappa_r^{-m}\mu_{rn} \quad (m \to \infty), \tag{46}$$

where

$$I_n \mu_{rn} = \int_0^1 \rho_0 f_n\{-2\lambda_n M_1 f_n \phi_r + \beta M_1(\phi_r f_n)' + \beta N^2 \phi_r' f_n'\} \, \mathrm{d}z.$$

So, finally, combining (42) with (44)–(46) gives

$$G_m(z) \sim (2m-1)! \sum_{\substack{r=1\\ r\neq n}}^{\infty} C_r(-1)^m \kappa_r^{-m} \phi_r(z) + (2m-3)! \frac{12}{\mu} f_n(z) \sum_{\substack{r=1\\ r\neq n}}^{\infty} C_r(-1)^m \kappa_r^{-m} \mu_{rn} \quad (m \to \infty). \tag{47}$$

4.2. *Borel summation*

It is clear from (47) that the series (32) is divergent. Nevertheless, as already remarked, following Pomeau *et al.* (1988), it is useful to attempt to 'sum' (32) using the Borel-summation technique (Bender & Orszag 1978, §8.2). Essentially, this is Watson's lemma applied in reverse: rather than generating an asymptotic expansion of a known function having a given integral representation, here we seek an integral expression for a function that has the known asymptotic expansion (32). Accordingly, we write

$$\psi(\xi, z) = \int_0^{\infty} \mathrm{d}u \, \mathrm{e}^{-u} V(z, u/\xi), \tag{48}$$

where

$$V = \sum_{m=1}^{\infty} \frac{G_m(z)}{(2m)!} \left(\frac{u}{\xi}\right)^{2m}; \tag{49}$$

applying Watson's lemma in the usual way, it is easy to check that the asymptotic expansion for $|\xi| \gg 1$ of ψ, defined in (48), (49), indeed agrees with (32). But the main advantage of this approach is that now the Borel transform V is expressed in terms of a convergent series in (49) that can be used to locate its singularities in the complex u-plane. As will be seen, these singularities play an important part in generating exponentially small terms. More specifically, in view of (47), the asymptotic form for large m of the general term, V_m, in the series (49) is

$$V_m \sim \sum_{\substack{r=1\\ r\neq n}}^{\infty} C_r \phi_r(z) \frac{(-1)^m}{2m} \left(\frac{u^2}{\kappa_r \xi^2}\right)^m + \frac{3}{2\mu} f_n(z) \sum_{\substack{r=1\\ r\neq n}}^{\infty} C_r \mu_{rn} \frac{(-1)^m}{m^3} \left(\frac{u^2}{\kappa_r \xi^2}\right)^m \quad (m \to \infty).$$

Therefore, V has logarithmic singularities at

$$u = \pm u_r = \pm \mathrm{i} \kappa_r^{\frac{1}{2}} \xi \quad (r = 1, 2, \ldots; r \neq n), \tag{50}$$

and the local behaviour of V near each of these singularities is

$$V \sim -\tfrac{1}{2} C_r \phi_r(z) \ln\left(1 - \frac{u^2}{u_r^2}\right) - \frac{3\mu_{rn}}{4\mu} C_r f_n(z) \left(1 - \frac{u^2}{u_r^2}\right)^2 \ln\left(1 - \frac{u^2}{u_r^2}\right) \quad (u \to \pm u_r). \tag{51}$$

5. Matching

We are now prepared to discuss the matching of the long-wave expansion (12) with the inner solution (48). To be specific, we shall suppose first that there are no oscillations at the left-hand tail ($X \to -\infty$) of the KdV solitary wave (15), (19) and determine the amplitude of possible oscillations at the right-hand tail ($X \to \infty$) through matching. Then, solitary-wave solutions that are symmetric about $X = 0$

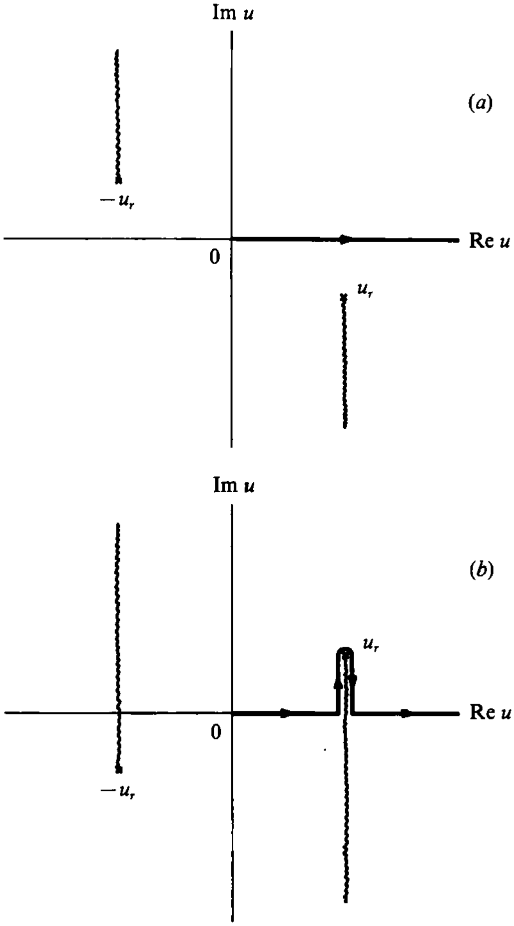

FIGURE 3. Position of the singularities of V at $u = \pm u_r = \pm i\kappa_r^{\frac{1}{2}}\xi$ (with $\kappa_r > 0$) and the corresponding branch cuts in the complex u-plane. (*a*) ξ lies in the left-hand matching region, $\mathrm{Re}\,\xi \rightarrow -\infty$, $\mathrm{Im}\,\xi < 0$, and the cuts do not interfere with the integration path along the positive real u-axis. (*b*) ξ lies in the right-hand matching region, $\mathrm{Re}\,\xi \rightarrow \infty$, $\mathrm{Im}\,\xi < 0$, and the integration path is deformed around the singularity at $u = u_r$ in order to avoid crossing the cut.

will be considered. The question of the existence of these as exact steady-state solutions is discussed later in this section.

As already noted in (51), the integrand in (48) has logarithmic singularities at $u = \pm u_r$, so suitable branch cuts need to be introduced. To ensure that (48) matches with (12) in the left matching region ($\mathrm{Re}\,\xi \rightarrow -\infty, \mathrm{Im}\,\xi < 0$) – as required by the assumption that no oscillations appear at the left tail – the u-plane is cut with branch cuts that extend away from the real u-axis (figure 3*a*). Thus, when ξ lies in the left-hand matching region, the path of integration in (48) does not interfere with the cuts, and, using Watson's lemma, it follows that (48) indeed matches with (32), the inner limit of (12). Now, as ξ is varied, the singularities (50) as well as the corresponding branch cuts shift in the complex u-plane, and, in fact, some of them may have crossed the positive real u-axis when ξ lies in the right matching region ($\mathrm{Re}\,\xi \rightarrow \infty, \mathrm{Im}\,\xi < 0$). If that happens, the path of integration in (48) has to be deformed around these singularities to avoid crossing the cuts (figure 3*b*). In deforming the integration path, however, an extra contribution to the integral in (48) arises from each singularity.

These turn out to be exponentially small terms that lie beyond all orders of the asymptotic expansion (32), and, upon matching, give rise to oscillations at the right-hand tail.

More specifically, it follows from (50) that singularities at $u = \pm u_r$ with $\kappa_r < 0$ never cross the path of integration in (48) as $\operatorname{Re}\xi$ is varied from $-\infty$ to ∞ ($\operatorname{Im}\xi < 0$), so they play no role in the matching. On the other hand, if the eigenvalue problem (10) has positive eigenvalues κ_r ($r = 1, \dots, n-1$) – as already remarked this is the case if $n > 1$ – it is easy to check that the corresponding singularities at $u = u_r$ have crossed the positive real u-axis when ξ lies in the right-hand matching region (figure 3*b*). Then, deforming the integration path around these singularities, taking into account the dominant singularity in (51) (the contribution of the second term in (51) to the solitary-wave tail turns out to be subdominant), and using Watson's lemma to expand the integral in (48), yields

$$\psi \sim \sum_{m=1} G_m(z)\,\xi^{-2m} + \mathrm{i}\pi \sum_{r=1}^{n-1} C_r\,\phi_r(z) \int_{u_r}^{\infty} \mathrm{e}^{-u}\,\mathrm{d}u \quad (\operatorname{Re}\xi \to \infty, \operatorname{Im}\xi < 0).$$

Evaluating the integrals above and recalling the definition of ξ in (29), it is clear that the singularity at $u = u_{n-1}$ makes the dominant contribution as $\epsilon \to 0$ (since $\kappa_1 > \kappa_2 > \dots > \kappa_{n-1} > 0$). Hence,

$$\psi \sim \sum_{m=1} G_m(z)\,\xi^{-2m} + \mathrm{i}\pi C_{n-1}\,\phi_{n-1}(z) \exp\left\{-\pi\frac{\kappa_{n-1}^{\frac{1}{2}}}{2\gamma\epsilon}\right\} \exp\left\{-\mathrm{i}\kappa_{n-1}^{\frac{1}{2}}\frac{X}{\epsilon}\right\}$$
$$(\operatorname{Re}\xi \to \infty, \operatorname{Im}\xi < 0). \quad (52)$$

This is the outer limit of the inner solution $\psi(\xi, z)$ in the right-hand matching region, near the singularity of the long-wave expansion (12) at $X = X_0$. As already mentioned, the second term in (52), arising from the singularity at $u = u_{n-1}$ in (48), is exponentially small.

Following the same procedure, an expression for ψ, similar to (52), can be derived near the singularity of the long-wave expansion (12) at $X = -X_0$. In terms of $X = -X_0 + \epsilon\hat{\xi}$, one has

$$\psi \sim \sum_{m=1} G_m(z)\,\hat{\xi}^{-2m} - \mathrm{i}\pi C_{n-1}\,\phi_{n-1}(z) \exp\left\{-\pi\frac{\kappa_{n-1}^{\frac{1}{2}}}{2\gamma\epsilon}\right\} \exp\left\{\mathrm{i}\kappa_{n-1}^{\frac{1}{2}}\frac{X}{\epsilon}\right\}$$
$$(\operatorname{Re}\hat{\xi} \to \infty, \operatorname{Im}\hat{\xi} > 0). \quad (53)$$

Now, the asymptotic behaviour of $\psi(X, z; \epsilon^2)$ at the right-hand tail ($X \to \infty$) of the solitary wave (15), (19) should be consistent with both inner expansions (52), (53). So, combining (12) with (52), (53), the appropriate asymptotic expansion at the right-hand tail is

$$\psi \sim \epsilon^2(\psi_0 + \epsilon^2\psi_1 + \dots) + 2\pi C_{n-1}\,\phi_{n-1}(z) \exp\left\{-\pi\frac{\kappa_{n-1}^{\frac{1}{2}}}{2\gamma\epsilon}\right\} \sin \kappa_{n-1}^{\frac{1}{2}} x. \quad (54)$$

Note that in the right-hand matching region near $X = \pm X_0$

$$\sin \kappa_{n-1}^{\frac{1}{2}} x \sim \pm\tfrac{1}{2}\mathrm{i} \exp\left\{\mp \mathrm{i}\kappa_{n-1}^{\frac{1}{2}}\frac{X}{\epsilon}\right\},$$

so that (54) matches with (52), (53), as required.

According to the asymptotic solution (54), exponentially small short-wave oscillations of mode $n-1$ necessarily appear at the right-hand tail of a KdV solitary wave of mode $n > 1$, assuming that the left-hand tail is free of oscillations. However, this is impossible for an exact steady-state solution since it implies a non-zero energy flux at the right-hand tail, but a zero energy flux at the left-hand tail. Hence our argument should be construed as establishing the non-existence of such exact asymmetric steady-state solutions. Note that Boyd (1991) conjectured from his numerical results that there are no asymmetric steady solitary-wave solutions of the fifth-order KdV equation. Instead, we expect that a localized initial disturbance will give rise to 'solitary-like' waves of mode $n > 1$ that decay slowly owing to the radiation of small-amplitude mode-$(n-1)$ short waves downstream. (In the analogous problem of equatorial Rossby waves, such a radiation-damping mechanism of higher-mode solitary waves has been found to exist according to the numerical model of Williams & Wilson (1988).) This seems to explain the appearance of mode-1 oscillations behind the mode-2 main disturbance in figure 2 and in the laboratory experiments of Davis & Acrivos (1967). In simpler problems, where the magnitude of the radiated oscillatory tail is algebraically small, the resulting decay of the solitary-wave amplitude has been analysed using perturbation theory (Akylas 1991), but this question will not be pursued here.

The same technique can be also used to discuss the asymptotic behaviour at the tails of solitary-wave solutions that are symmetric about $X = 0$, $\psi(X, z; \epsilon^2) = \psi(-X, z; \epsilon^2)$. The symmetry condition implies that the inner solutions near $X = \pm X_0$ have to be real-valued when the corresponding inner variables are purely imaginary. To satisfy this requirement, note that the singularities at $u = u_r$ with $\kappa_r > 0$ $(r = 1, \ldots, n-1)$ of the integrand in (48) are on the positive real u-axis when $\mathrm{Re}\,\xi = 0$. Accordingly, (48) is replaced with

$$\psi(\xi, z) = \frac{1}{2}\int_{\mathscr{C}_+} \mathrm{d}u\, \mathrm{e}^{-u} V(z, u/\xi) + \frac{1}{2}\int_{\mathscr{C}_-} \mathrm{d}u\, \mathrm{e}^{-u} V(z, u/\xi), \tag{55}$$

where the integration path $\mathscr{C}_+$ is deformed to pass above the singularities (and the corresponding cuts extend below the real u-axis), while $\mathscr{C}_-$ is deformed to pass below the singularities (and the corresponding cuts extend above the real u-axis). Hence, when $\mathrm{Re}\,\xi = 0$, the contributions from the singularities cancel out, so that the outer limit $(\mathrm{Im}\,\xi \to -\infty, \mathrm{Re}\,\xi = 0)$ of (55) coincides with (32), and ψ is indeed real-valued. Now, when ξ is in the right-hand matching region, only the singularities in the integrand of the first integral in (55) contribute exponentially small terms to the outer limit of ψ; the singularities in the integrand of the second integral do not interfere with the integration path $\mathscr{C}_-$. Hence, the amplitude of the oscillations at the right-hand tail is one half of that in (54):

$$\psi \sim \epsilon^2(\psi_0 + \epsilon^2\psi_1 + \ldots) + \pi C_{n-1}\,\phi_{n-1}(z) \exp\left\{-\pi\frac{\kappa_{n-1}^{\frac{1}{2}}}{2\gamma\epsilon}\right\} \sin \kappa_{n-1}^{\frac{1}{2}} x \quad (X > 0). \tag{56a}$$

Similarly, when ξ is in the left-hand matching region, only the second integral in (55) contributes exponentially small terms. Taking into account the fact that the integration path $\mathscr{C}_-$ now passes around the singularities in the counterclockwise direction, one then obtains at the left-hand tail

$$\psi \sim \epsilon^2(\psi_0 + \epsilon^2\psi_1 + \ldots) - \pi C_{n-1}\,\phi_{n-1}(z) \exp\left\{-\pi\frac{\kappa_{n-1}^{\frac{1}{2}}}{2\gamma\epsilon}\right\} \sin \kappa_{n-1}^{\frac{1}{2}} x \quad (X < 0). \tag{56b}$$

As expected, the asymptotic behaviours (56) are consistent with the assumed symmetry about the main peak at $X = 0$. The resulting wave is similar to the symmetric capillary–gravity waves with oscillatory tails computed by Hunter & Vanden-Broeck (1983) in the low-surface-tension regime. These symmetric solitary waves with oscillatory tails are consistent on energetic grounds, and we conjecture that they are exact solutions of the steady-state equations. However, because the energy flux in the tails has the same sign at both ends, presumably they can not be excited from a localized initial condition.

Expressions (54), (56) for the asymptotic behaviour at the tails of a KdV solitary wave of mode $n > 1$ are the main results of the asymptotic theory. It is noteworthy that, as suggested by the intuitive arguments of §1, the short-wave modes $\phi_r(z)$ $(r = 1, \ldots, n-1)$ with the same phase speed as the solitary wave arise naturally in the asymptotic analysis. Furthermore, it has been established that the $(n-1)$-mode makes the dominant contribution to the oscillatory tails. (In special situations, as in Tung *et al.* (1982), symmetry considerations imply that the $(n-1)$-mode is not excited, in which case the $(n-2)$-mode is expected to dominate for $n > 2$.)

Before proceeding to a discussion in §6 of the value of the constant C_{n-1}, it should be pointed out that the symmetric solution constructed in (56*a*, *b*) is in fact only a member of a one-parameter family of solitary waves with oscillatory tails. This arises because one may add to the inner solution (55) the subdominant term

$$\frac{1}{2}\sum_{r=1}^{n-1} B_r\,\phi_r(z)\exp\{-\mathrm{i}\kappa_r^{\frac{1}{2}}\xi\}, \tag{57}$$

where B_r are real constants so that (55) remains real-valued when $\operatorname{Re}\xi = 0$, as required. Hence, keeping only the dominant term in (57) for which $r = n-1$, we must add to the expressions (56*a*, *b*) the term

$$B_{n-1}\,\phi_{n-1}(z)\exp\left\{-\pi\frac{\kappa_{n-1}^{\frac{1}{2}}}{2\gamma\epsilon}\right\}\cos\kappa_{n-1}^{\frac{1}{2}}x. \tag{58}$$

Combining (56) with (58), the oscillatory tails of symmetric solitary waves then have the form

$$B\phi_{n-1}(z)\exp\left\{-\pi\frac{\kappa_{n-1}^{\frac{1}{2}}}{2\gamma\epsilon}\right\}\sin\left(\kappa_{n-1}^{\frac{1}{2}}|x|+\chi\right), \tag{59}$$

where the amplitude B and the phase χ are given by

$$B = (B_{n-1}^2+\pi^2C_{n-1}^2)^{\frac{1}{2}}, \quad \tan\chi = \frac{B_{n-1}}{\pi C_{n-1}}.$$

Equivalently, $B\cos\chi = \pi C_{n-1}$ defines the phase χ in terms of the amplitude B. The result that symmetric internal solitary waves with oscillatory tails occur as a one-parameter family is consistent with the theoretical results of Beale (1991) for gravity–capillary waves, and Boyd (1991) and Amick & Toland (1992) for the fifth-order KdV equation.

The amplitude of the oscillatory tails in (54), (59) depends on the constant C_{n-1} which, as already remarked, cannot be obtained from asymptotic analysis alone. Returning to (33), (34), to determine C_{n-1}, it is necessary to solve a sequence of boundary-value problems for $G_m(z)\,(m > 1)$, and match with the asymptotic behaviour (41) of the solution for large values of m. These are linear, inhomogeneous, two-point boundary-value problems and, in principle, can be solved numerically in a straightforward manner. However, as noted in §4.1, the (known) forcing terms in the equation for $G_m(z)$, that arise from the nonlinear terms in (33), involve multiple convolution sums of $G_1(z), \ldots, G_{m-1}(z)$ and become very cumbersome to evaluate as m increases. This is not unexpected because the inner equation (31) is fully nonlinear; even though the oscillatory tails are dominated by the linear mode ϕ_{n-1} according to (54), (59), the constant C_{n-1} and, hence, the precise value of the oscillation amplitude depends on all nonlinear terms. To illustrate the procedure for calculating C_{n-1}, two simple examples of density stratification, for which the nonlinear terms in (31) are only quadratic, are discussed below.

6. Examples

For simplicity, the density stratification is assumed to be such that $N^2(z)$ varies linearly with depth:

$$N^2(z) = 1+bz \quad (0 \leqslant z \leqslant 1),$$

where $b > 0$ is a constant. Then, the Taylor-series expansion (14) terminates at the linear term, $M_j = 0\,(j \geqslant 2)$, and $G_m(z)\,(m \geqslant 1)$ satisfy the simplified equation sequence (35). Furthermore, to avoid calculating cubic terms in (35), two particular cases will be considered:

(i) $b = 0, \quad \beta \neq 0$.

(ii) $b \neq 0, \quad \beta = 0$.

Case (i) corresponds to exponential density variation,

$$\rho_0(z) = \exp(-\beta z) \quad (0 \leqslant z \leqslant 1),$$

and the solutions to the eigenvalue problems (8), (10) are known in closed form:

$$\lambda_n = n^2\pi^2 + \tfrac{1}{4}\beta^2, \quad f_n(z) = \exp(\tfrac{1}{2}\beta z)\sin n\pi z \quad (n = 1, 2, \ldots); \tag{60a}$$

$$\kappa_r = (n^2 - r^2)\pi^2, \quad \phi_r(z) = \exp(\tfrac{1}{2}\beta z)\sin r\pi z \quad (r = 1, 2, \ldots). \tag{60b}$$

In (ii), the Boussinesq approximation ($\beta = 0$) implies that ρ_0 is constant ($\rho_0 = 1$). The eigenvalue problems (8), (10) have to be solved numerically in this case. Table 1 lists the first three eigenvalues for various values of b. The corresponding eigenfunctions are normalized so that the normalization constants in (9), (11) are equal to $\frac{1}{2}$.

The constant C_{n-1} is computed by solving the sequence of inhomogeneous boundary-value problems (35), (34) for $G_m(z)\,(m > 1)$ numerically, with $G_1(z)$ given by (36). As in (37), for each $m > 1$, the solution is expressed as the superposition of an inhomogeneous solution, $Q_m(z)$, plus a multiple of the homogeneous solution $f_n(z)$:

$$G_m(z) = Q_m(z) + \gamma_m f_n(z),$$

and the constant γ_m is found by imposing the compatibility condition

$$\int_0^1 \rho_0 R_{m+1} f_n \,\mathrm{d}z = 0$$

on the forcing term R_{m+1}, of the next boundary-value problem in the sequence. In implementing this procedure, since $G_m(z)$ grows rapidly as m increases, it proves convenient to make the substitution

$$G_m(z) = (-1)^m \kappa_{n-1}^{-m}\, U_m(z)\,(2m-1)!$$

and work with $U_m(z)$. Then, in view of (41),

$$U_m(z) \sim C_{n-1}\phi_{n-1} + \sum_{\substack{r=1 \\ r \neq n,\, n-1}}^{\infty} C_r \left(\frac{\kappa_{n-1}}{\kappa_r}\right)^m \phi_r(z) \quad (m \to \infty), \tag{61}$$

b	λ_1	λ_2	λ_3	κ_1	κ_3
1.0	6.55	26.5	59.7	30.6	−49.3
1.5	5.60	22.8	51.4	31.2	−49.2
2.0	4.88	20.0	45.2	31.8	−49.2
2.5	4.33	17.8	40.3	32.3	−49.2
3.0	3.89	16.1	36.4	32.7	−49.1

TABLE 1. First few eigenvalues of problems (8), (10) in case (ii) of density stratification for various values of the parameter b, and $n = 2$. Note that $\kappa_2 = 0$ for all b (corresponding to the long-wave mode).

(i)		(ii)	
β	C_1	b	C_1
1.0	0.14	1.0	−8.1
1.5	−0.33	1.5	1.8
2.0	0.27	2.0	5.7
2.5	0.54	2.5	5.2
3.0	−0.12	3.0	3.9

TABLE 2. Values of the constant C_1 for various values of the parameters β, b and $n = 2$ in cases (i), (ii) of density stratification.

so that the constant C_{n-1} can be estimated from the numerical solution for $U_m(z)$, for large values of m, by projecting $U_m(z)$ on $\phi_{n-1}(z)$, using the orthogonality property (11).

Numerically computed values of the constant C_{n-1} are listed in table 2 for certain values of the parameters β, b in (i), (ii) and $n = 2$. For this value of n, according to (60) and table 1, only one eigenvalue, κ_1, of the problem (10) is positive, and the amplitude of the oscillatory tails is proportional to C_1. In the numerical solution of the boundary-value problems (35), (34), the asymptotic solution (61) has been reached and convergence of the value of C_1 to two significant figures obtains, typically, for values of m larger than about 80. As expected, C_1 is non-zero in general, and mode-1 short waves certainly appear at the tails of a mode-2 solitary wave. However, it is clear from table 1 that C_1 can be zero for special values of the parameters. In such a case, the amplitude of these oscillations may vanish, at least to leading order in the expansion. It would be interesting to have independent confirmation of the predictions of the asymptotic theory by fully numerical computation and laboratory experimental observations of solitary waves of mode $n > 1$. In fact, in very recent numerical work, Vanden-Broeck (1992) has confirmed that mode-2 solitary waves can develop mode-1 oscillatory tails.

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